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The multicomponent Eckhaus equation

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Abstract. The 'n-component Eckhaus equation',

$$\mathrm{i}\psi_t + \mathbf{A}\psi_{xx} + (\psi, \mathbf{C}\psi)_x\mathbf{A}\psi + 2(\psi, \mathbf{C}\psi)\mathbf{A}\psi_x + [(\psi_x, \mathbf{CA}\psi) - (\psi, \mathbf{CA}\psi_x)]\psi + (\psi, \mathbf{C}\psi)^2\mathbf{A}\psi = 0$$

is C-integrable. Here the dependent variable $\psi \equiv \psi(x, t)$ is an n-vector, and $\mathbf{A}$ and $\mathbf{C}$ are two constant $(n \times n)$-matrices restricted by the condition that both $\mathbf{C}$ and $\mathbf{CA}$ be Hermitian. Some special cases, and some explicit solutions, are displayed.

1. Introduction

Some years ago it was noted that the *nonlinear* evolution partial differential equation

$$\mathrm{i}\psi_t + \psi_{xx} + \psi[|\psi|^4 + 2(|\psi|^2)_x] = 0 \qquad \psi \equiv \psi(x, t) \tag{1.1}$$

has a *universal* character and is *C-integrable*, [1], being linearizable via the change of dependent variable

$$\varphi(x, t) = \psi(x, t)\exp\left[\int_{-\infty}^{x} \mathrm{d}x'|\psi(x', t)|^2\right] \tag{1.2a}$$

$$\psi(x, t) = \varphi(x, t)\Big/\left[1 + 2\int_{-\infty}^{x} \mathrm{d}x'|\varphi(x', t)|^2\right]^{1/2} \tag{1.2b}$$

which transforms (1.1) into the *linear* Schrödinger equation

$$\mathrm{i}\varphi_t + \varphi_{xx} = 0 \qquad \varphi \equiv \varphi(x, t). \tag{1.3}$$

The importance of the universal character of (1.1) was subsequently elaborated in the context of a detailed explanation of the remarkable fact, that certain nonlinear evolution PDEs are both widely applicable and integrable [2].

Since the original idea which underpins the universal character of (1.1) is due to Eckhaus, this equation was called the 'Eckhaus equation' in a paper which used the *direct* and *inverse* transformations (1.2) to analyse the detailed behaviour of various solutions of (1.1), thereby explicitly displaying the mechanism that underlies certain remarkable 'solitonic' phenomena (such as the predominantly 'elastic' character of solitonic collisions) [3].

Subsequently the Eckhaus equation (1.1) has been extended to include an external potential [4], and it has been generalized to an $(N + 1)$-dimensional context [5] (N space

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variables and one time variable; the original Eckhaus equation (1.1) refers of course to a $(1 + 1)$-dimensional context). Some graphs of soliton-like solutions of the $(2 + 1)$-dimensional Eckhaus equation in an external oscillator potential have also been published [6].

The purpose and scope of this paper is to exhibit the natural extension of the Eckhaus equation (1.1) to the *multicomponent* case, including the *direct* and *inverse* transformations that demonstrate its *C-integrability*. We also exhibit various special cases and reductions, show how to solve the initial-value ('Cauchy') problem, and analyse in detail various examples of solitonic behaviour, including the display of (single) solitons which move with non-constant speeds and may even boomerang back. Extensions to include an external potential [4] and to a multidimensional context [5, 7] are easy.

Note that, in writing the direct and inverse transformations (1.2), we have implicitly assumed that both $\varphi(x,t)$ and $\psi(x,t)$ vanish, as $x \to -\infty$, sufficiently fast to guarantee that the integrals on the right-hand sides of these equations converge. These transformations are particularly appropriate for studing the Eckhaus equation (1.1) on the whole line, with vanishing boundary conditions at $x = -\infty$. For the treatment of more general cases, including boundary-value problems and involving transformations of type (1.2) with a finite lower limit of integration, the interested reader is referred to [2, 8].

Let us finally recall that the integrable character of the nonlinear evolution equation (1.1) was known [9] before its universality was recognized [1, 2], and let us reiterate that in this paper our presentation is limited to exhibiting the multicomponent Eckhaus equation (see (2.1) below) and demonstrating its C-integrability. The general technique which underpins these results has been amply illustrated in previous papers [9, 2, 10, 5, 7, 11]; indeed the last of these papers already presents (a special case of) the multicomponent Eckhaus equation.

2. The multicomponent Eckhaus equation

The *multicomponent* (or, equivalently, *n-vector*) Eckhaus equation reads as follows

$$\begin{aligned} &\mathrm{i}\boldsymbol{\psi}_t + \mathbf{A}\boldsymbol{\psi}_{xx} + (\boldsymbol{\psi}, \mathbf{C}\boldsymbol{\psi})_x\mathbf{A}\boldsymbol{\psi} + 2(\boldsymbol{\psi}, \mathbf{C}\boldsymbol{\psi})\mathbf{A}\boldsymbol{\psi}_x \\ &+[(\boldsymbol{\psi}_x, \mathbf{CA}\boldsymbol{\psi}) - (\boldsymbol{\psi}, \mathbf{CA}\boldsymbol{\psi}_x)]\boldsymbol{\psi} + (\boldsymbol{\psi}, \mathbf{C}\boldsymbol{\psi})^2\mathbf{A}\boldsymbol{\psi} = 0 \end{aligned} \tag{2.1a}$$

or, equivalently (see below),

$$\mathrm{i}\boldsymbol{\psi}_t + \mathbf{A}\boldsymbol{\psi}_{xx} + 2F\mathbf{A}\boldsymbol{\psi}_x + [G + (F_x + F^2)\mathbf{A}]\boldsymbol{\psi} = 0 \tag{2.1b}$$

where the scalar functions F and G are

$$F = (\boldsymbol{\psi}, \mathbf{C}\boldsymbol{\psi}) \qquad G = 2\mathrm{i}\,\mathrm{Im}(\boldsymbol{\psi}_x, \mathbf{CA}\boldsymbol{\psi}). \tag{2.1c}$$

Here and below the dependent variable $\boldsymbol{\psi} \equiv \boldsymbol{\psi}(x,t)$ is an n-vector; for n-vectors we employ the scalar-product notation

$$(\boldsymbol{u}, \boldsymbol{v}) \equiv \sum_{m=1}^{n} u_m^* v_m \tag{2.2}$$

and $\mathbf{A}$ and $\mathbf{C}$ are two constant $(n \times n)$-matrices satisfying the (n-matrix) conditions

$$(\mathbf{CA})^+ = \mathbf{CA} \tag{2.3}$$

$$\mathbf{C}^+ = \mathbf{C}, \tag{2.4}$$

where $\mathbf{M}^+$ is the Hermitian conjugate of $\mathbf{M}$ so that

$$(\mathbf{M}^+)_{jk} \equiv M_{kj}^*. \tag{2.5}$$

Let us display some special cases.

If the matrix $\mathbf{A}$ is Hermitian, $\mathbf{A}^+ = \mathbf{A}$, then $\mathbf{A}$ and $\mathbf{C}$ are two commuting Hermitian matrices (see (2.3) and (2.5)) which, with no loss of generality, can be assumed to be real and diagonal,

$$\mathbf{A} = \mathrm{diag}(a_m) \qquad a_m = a_m^* \tag{2.6a}$$

$$\mathbf{C} = \mathrm{diag}(c_m) \qquad c_m = c_m^*. \tag{2.6b}$$

In this case, (2.1) becomes

$$\mathrm{i}\psi_{j,t} + a_j\psi_{j,xx} + \sum_{m=1}^{n}\{c_m[(|\psi_m|^2)_x(a_j + a_m)\psi_j + 2\psi_m^*(a_j\psi_m\psi_{j,x} - a_m\psi_j\psi_{m,x})]\} + \left(\sum_{m=1}^{n} c_m|\psi_m|^2\right)^2 a_j\psi_j = 0. \tag{2.7}$$

Note that, via the rescaling $\psi_m \to \psi_m' = |c_m|^{-1/2}\psi_m$, the constants c_m can be eliminated, except for their signs. Of course, for $n = 1$, (2.6) reproduces (1.1) (up to trivial rescalings). For $n = 2$, it reads:

$$\mathrm{i}\psi_{1,t} + a_1\psi_{1,xx} + 2a_1(c_1|\psi_1|^2 + c_2|\psi_2|^2)\psi_{1,x} + 2a_1c_1\psi_1^2\psi_{1,x}^* + (a_1 - a_2)c_2\psi_1\psi_2^*\psi_{2,x} + (a_1 + a_2)c_2\psi_1\psi_2\psi_{2,x}^* + (c_1|\psi_1|^2 + c_2|\psi_2|^2)^2a_1\psi_1 = 0 \tag{2.7a}$$

$$\mathrm{i}\psi_{2,t} + a_2\psi_{2,xx} + 2a_2(c_1|\psi_1|^2 + c_2|\psi_2|^2)\psi_{2,x} + 2a_2c_2\psi_2^2\psi_{2,x}^* - (a_1 - a_2)c_1\psi_2\psi_1^*\psi_{1,x} + (a_1 + a_2)c_1\psi_1\psi_2\psi_{1,x}^* + (c_1|\psi_1|^2 + c_2|\psi_2|^2)^2a_2\psi_2 = 0. \tag{2.7b}$$

Let us now consider a different case in which $\mathbf{A}$ is diagonalizable (indeed, without loss of generality, diagonal) but not Hermitian; and, for the sake of simplicity, let us display the special case with $n = 2$, namely

$$\mathbf{A} = \begin{pmatrix} h & 0 \\ 0 & h^* \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 0 & g \\ g^* & 0 \end{pmatrix} \tag{2.8}$$

with h and g two *arbitrary complex constants*. Note that this choice satisfies (2.3) and (2.5).

Then (2.1) reads

$$\mathrm{i}\psi_{1,t} + h\psi_{1,xx} + 2h\psi_1\,\mathrm{Re}(g\psi_1^*\psi_2)_x + 4h\psi_{1,x}\,\mathrm{Re}(g\psi_1^*\psi_2) + 2\mathrm{i}\psi_1\,\mathrm{Im}(hg^*\psi_1\psi_{2,x}^* + h^*g\psi_{1,x}^*\psi_2) + 4h\psi_1[\mathrm{Re}(g\psi_1^*\psi_2)]^2 = 0 \tag{2.9a}$$

$$\mathrm{i}\psi_{2,t} + h^*\psi_{2,xx} + 2h^*\psi_2\,\mathrm{Re}(g\psi_1^*\psi_2)_x + 4h^*\psi_{2,x}\,\mathrm{Re}(g\psi_1^*\psi_2) + 2\mathrm{i}\psi_2\,\mathrm{Im}(hg^*\psi_1\psi_{2,x}^* + h^*g\psi_{1,x}^*\psi_2) + 4h^*\psi_2[\mathrm{Re}(g\psi_1^*\psi_2)]^2 = 0\ . \tag{2.9b}$$

Finally, let us display a third case, again with $n = 2$ for the sake of simplicity, in which $\mathbf{A}$ is not diagonalizable (and, without loss of generality, in Jordan form):

$$\mathbf{A} = \begin{pmatrix} 1 & a\exp(\mathrm{i}\alpha) \\ 0 & 1 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 0 & b\exp(\mathrm{i}\alpha) \\ b\exp(-\mathrm{i}\alpha) & c \end{pmatrix} \tag{2.10}$$

with a, b, c and α four *arbitrary real constants*. Note that this choice satisfies (2.3) and (2.5).

Then (2.1) reads

$$\mathrm{i}\psi_{1,t} + \psi_{1,xx} + a\exp(\mathrm{i}\alpha)\psi_{2,xx} + F_x[\psi_1 + a\exp(\mathrm{i}\alpha)\psi_2] + 2F[\psi_1 + a\exp(\mathrm{i}\alpha)\psi_2]_x + G\psi_1 + F^2[\psi_1 + a\exp(\mathrm{i}\alpha)\psi_2] = 0 \tag{2.11a}$$

$$\mathrm{i}\psi_{2,t} + \psi_{2,xx} + 2F\psi_{2,x} + (G + F_x + F^2)\psi_2 = 0 \tag{2.11b}$$

$$F = 2b\,\mathrm{Re}[\exp(\mathrm{i}\alpha)\psi_1^*\psi_2] + c|\psi_2|^2 \tag{2.11c}$$

$$G = 2\mathrm{i}\{b\,\mathrm{Im}[\exp(\mathrm{i}\alpha)(\psi_2\psi_{1,x}^* - \psi_1^*\psi_{2,x})] + (ab + c)\,\mathrm{Im}(\psi_2\psi_{2,x}^*)\}. \tag{2.11d}$$

We close this section with a brief discussion of the problem of reducing the number n of components of the vector $\boldsymbol{\psi}$ that satisfies the multicomponent Eckhaus equation (2.1). A trivial reduction is obtained by requiring that $\boldsymbol{\psi}$ is a linear combination of $p < n$ linearly independent eigenvectors of $\mathbf{A}$, say $\mathbf{A}\boldsymbol{v}^{(j)} = a_j \boldsymbol{v}^{(j)}$, $j = 1, 2, \ldots, p$, $\boldsymbol{\psi} = \sum_{j=1}^{p} \psi_j \boldsymbol{v}^{(j)}$. One then reobtains (2.7), with n replaced by p. In the simplest case $p = 1$, after obvious rescalings, this reduces, of course, to the Eckhaus equation (1.1).

A less trivial reduction obtains, by requiring that the solution $\boldsymbol{\psi}$ of (2.1) solves the algebraic equation,

$$\boldsymbol{\sigma}\boldsymbol{\psi} = \boldsymbol{\psi}^* \tag{2.12}$$

where the $n \times n$ matrix $\boldsymbol{\sigma}$ is such that

$$\boldsymbol{\sigma}\mathbf{A} + \mathbf{A}^*\boldsymbol{\sigma} = 0 \tag{2.13}$$

in order to guarantee that equations (2.1) and (2.12) are compatible. Indeed, reduction equation (2.12) implies a relation between the entries of $\boldsymbol{\psi}$, which reduces the number of independent fields. Below we display two simple instances of this reduction technique for $n = 2$.

Let us first consider the two-vector Eckhaus equation (2.7) with $a_1 \equiv a = -a_2$; then the matrix $\boldsymbol{\sigma}$ which satisfies the condition (2.13) is easily found to be

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 1/\lambda \\ \lambda^* & 0 \end{pmatrix} \tag{2.14}$$

with λ an arbitrary complex constant; and the reduction equation (2.12) implies $\psi_1 \equiv \psi, \psi_2 = \lambda\psi^*$. In this way, the two coupled equations (2.7) reduce to the scalar Eckhaus equation (1.1), after obvious rescalings of the dependent and independent variables.

Next consider the two-vector Eckhaus equation (2.9) with $h = \mathrm{i}d, d = d^*$; then the matrix $\boldsymbol{\sigma}$ in (2.13) is the unit matrix

$$\boldsymbol{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.15}$$

and the reduction equation (2.12) coincides with the reality condition $\psi_j^* = \psi_j$, $j = 1, 2$. In this case, system (2.9) becomes real, and it reads

$$\psi_{1t} + d\{\psi_{1xx} + 4g\psi_1[(\psi_1\psi_2)_x + g(\psi_1\psi_2)^2]\} = 0 \tag{2.16a}$$

$$\psi_{2t} - d\{\psi_{2xx} + 4g\psi_2[(\psi_1\psi_2)_x + g(\psi_1\psi_2)^2]\} = 0 \tag{2.16b}$$

where g and d are arbitrary real constants (they can both be set to unity by appropriate rescalings).

We finally note that, in contrast, no reduction of type (2.12) exists for the system (2.11) (with $a \neq 0$).

3. Derivation and technique of solution

Let the n-vector $\boldsymbol{\varphi}(x, t)$ satisfy the (linear one-dimensional) Schrödinger equation

$$\mathrm{i}\boldsymbol{\varphi}_t + \mathbf{A}\boldsymbol{\varphi}_{xx} = 0. \tag{3.1}$$

It is then clear that the (real; see (2.2) and (2.5)) 'charge density' $\rho(x, t)$,

$$\rho = (\boldsymbol{\varphi}, \mathbf{C}\boldsymbol{\varphi}) \tag{3.2}$$

satisfies the 'charge conservation' equation

$$\rho_t = j_x \tag{3.3}$$

with the (real; see (2.2) and (2.3)) 'current' $j(x,t)$ defined as follows

$$j = \mathrm{i}[(\varphi, \mathbf{CA}\varphi_x) - (\varphi_x, \mathbf{CA}\varphi)] \tag{3.4a}$$

$$j = 2\,\mathrm{Im}(\varphi_x, \mathbf{CA}\varphi). \tag{3.4b}$$

Here we are of course using (2.3) and (2.5).

Now introduce the n-vector $\psi(x,t)$ via the position

$$\varphi = V\psi \tag{3.5}$$

where V is a real scalar function,

$$V^*(x,t) = V(x,t). \tag{3.6}$$

It is then easily seen that, if one defines new 'charges' and 'currents' $R(x,t)$ and $J(x,t)$ via the position

$$R = (\psi, \mathbf{C}\psi) \tag{3.7}$$

$$J = \mathrm{i}[(\psi, \mathbf{CA}\psi_x) - (\psi_x, \mathbf{CA}\psi)] \tag{3.8a}$$

$$J = 2\,\mathrm{Im}(\psi_x, \mathbf{CA}\psi) \tag{3.8b}$$

there holds (as consequence of the reality of V, see (3.6), and of course of the position (3.5)) the relation

$$\rho = V^2 R \tag{3.9a}$$

as well as

$$j = V^2 J. \tag{3.9b}$$

Introduce now the real scalar function $u(x,t)$ via the positions

$$u_x = \rho \tag{3.10a}$$

$$u_t = j \tag{3.10b}$$

whose compatibility is guaranteed by (3.3), and likewise the real scalar function $U(x,t)$ via the positions

$$U_x = R \tag{3.11a}$$

$$U_t = J. \tag{3.11b}$$

The compatibility of these last positions requires validity of the 'charge conservation' equation

$$R_t = J_x. \tag{3.12}$$

It is indeed easily seen that this equality is implied by (3.3) and (3.9), provided (as we hereafter assume)

$$V(x,t) = V[U(x,t)]. \tag{3.13}$$

We may now compute the evolution equation satisfied by $\psi(x,t)$. Indeed (3.5), (3.13) and (3.11b) entail $(V'(U) = \mathrm{d}V(U)/\mathrm{d}U)$

$$\varphi_t = V'(U)J\psi + V(U)\psi_t \tag{3.14a}$$

while (3.5), (3.13) and (3.11a) entail

$$\varphi_x = V'(U)R\psi + V(U)\psi_x \tag{3.14b}$$

$$\varphi_{xx} = V''(U)R^2\psi + V'(U)[R_x\psi + 2R\psi_x] + V(U)\psi_{xx}. \tag{3.14c}$$

Hence (3.1) yields

$$\mathrm{i}\boldsymbol{\psi}_t + \mathbf{A}\boldsymbol{\psi}_{xx} + [V'(U)/V(U)](\mathrm{i}J\boldsymbol{\psi} + R_x\mathbf{A}\boldsymbol{\psi} + 2R\mathbf{A}\boldsymbol{\psi}_x) + [V''(U)/V(U)]R^2\mathbf{A}\boldsymbol{\psi} = 0. \tag{3.15}$$

This evolution equation for $\boldsymbol{\psi}$ should be supplemented by (3.11), and of course by (3.7) and (3.8). Note that the choice of the function $V(U)$ remains our privilege. Now we make the simplest choice

$$V(U) = \exp(U) \tag{3.16}$$

whereby the U-dependence disappears from (3.15), which then coincides with the n-component Eckhaus equation (2.1).

Having thereby completed the derivation of the n-component Eckhaus equation, let us indicate the technique to solve it. The key is relation (3.5) among the n-vector $\boldsymbol{\psi}(x,t)$, which satisfies the n-component Eckhaus equation (2.1), and the n-vector $\boldsymbol{\varphi}(x,t)$, which satisfies the *linear* Schrödinger equation (3.1). Via (3.16), (3.11*a*) and (3.7), this formula reads

$$\boldsymbol{\varphi}(x,t) = \boldsymbol{\psi}(x,t)\exp\left[\int_{-\infty}^{x}\mathrm{d}x'(\boldsymbol{\psi}(x',t),\mathbf{C}\boldsymbol{\psi}(x',t))\right]. \tag{3.17a}$$

Hereafter we assume for simplicity that the field $\boldsymbol{\psi}(x,t)$ (hence $\boldsymbol{\varphi}(x,t)$ as well) vanish asymptotically as $x \to -\infty$ at least as $|x|^{\epsilon-1/2}$ with $\epsilon < 0$.

The inverse formula reads

$$\boldsymbol{\psi}(x,t) = \boldsymbol{\varphi}(x,t)\left[1 + 2\int_{-\infty}^{x}\mathrm{d}x'(\boldsymbol{\varphi}(x',t),\mathbf{C}\boldsymbol{\varphi}(x',t))\right]^{-1/2}. \tag{3.17b}$$

For completeness, we provide a derivation of this formula in the appendix.

Explicit solutions of the n-component Eckhaus equation (2.1) can now be easily constructed by inserting solutions $\boldsymbol{\varphi}(x,t)$ of the linear Schrödinger equation (3.1) into (3.17*b*).

The Cauchy problem for the n-component Eckhaus equation (2.1)—namely, the problem to evaluate $\boldsymbol{\psi}(x,t)$ from a given $\boldsymbol{\psi}(x,0)$—is solved by using (3.17*a*) to evaluate $\boldsymbol{\varphi}(x,0)$, by then solving the *linear* Schrödinger equation (3.1) in order to obtain $\boldsymbol{\varphi}(x,t)$, and finally by obtaining $\boldsymbol{\psi}(x,t)$ via (3.17*b*).

Let us also display the resolving formulae for the special cases displayed in the previous section. The linearizing formula (3.17*b*) corresponding to the two-component equation (2.7) reads

$$\psi_j(x,t) = \varphi_j(x,t)\left\{1 + 2\int_{-\infty}^{x}\mathrm{d}x'[c_1|\varphi_1(x',t)|^2 + c_2|\varphi_2(x',t)|^2]\right\}^{-1/2} \qquad j = 1,2 \tag{3.18}$$

where φ_j is a solution of the scalar Schrödinger equation $\mathrm{i}\varphi_{j,t} + a_j\varphi_{j,xx} = 0$. For system (2.9), solution (3.17*b*) reads

$$\psi_j(x,t) = \varphi_j(x,t)\left\{1 + 4\int_{-\infty}^{x}\mathrm{d}x'\,\mathrm{Re}[g\varphi_1^*(x',t)\varphi_2(x',t)]\right\}^{-1/2} \qquad j = 1,2 \tag{3.19}$$

with φ_1 and φ_2 solutions of the scalar equations $\mathrm{i}\varphi_{1,t} + h\varphi_{1,xx} = 0$ and, respectively, $\mathrm{i}\varphi_{2,t} + h^*\varphi_{2,xx} = 0$. And for the third system of equations (2.11), one obtains the solution

$$\psi_j(x,t) = \varphi_j(x,t)\left\{1 + 2c\int_{-\infty}^{x}\mathrm{d}x'|\varphi_2(x',t)|^2 \right.$$
$$\left. + 4b\int_{-\infty}^{x}\mathrm{d}x'\,\mathrm{Re}[\exp(\mathrm{i}\alpha)\varphi_1^*(x',t)\varphi_2(x',t)]\right\}^{-1/2} \qquad j = 1,2. \tag{3.20}$$

Here $\varphi_2(x,t)$ is a solution of the scalar Schrödinger equation $\mathrm{i}\varphi_{2,t}+\varphi_{2,xx}=0$, while $\varphi_1(x,t)$ is given by the formula

$$\varphi_1(x,t)=\tilde{\varphi}_1(x,t)+\mathrm{i}a\exp(\mathrm{i}\alpha)t\varphi_{2,xx}(x,t) \tag{3.21}$$

where $\tilde{\varphi}_1$ satisfies the same Schrödinger equation, $\mathrm{i}\tilde{\varphi}_{1,t}+\tilde{\varphi}_{1,xx}=0$, as φ_2. This formula follows from the triangularity of the two-component linear Schrödinger equation (3.1) with (2.10).

4. Examples of soliton solutions

In this section we construct some explicit solutions of the n-vector Eckhaus equation in the case $n=2$.

We first assume that both $\mathbf{A}$ and $\mathbf{C}$ are real, diagonal matrices. In this case $\psi_1(x,t)$ and $\psi_2(x,t)$ satisfy the two coupled nonlinear equations (2.7*a*) and (2.7*b*). Just for definiteness, and without loss of generality, we also assume the dispersion coefficients to be positive, $a_1>0$ and $a_2>0$.

Explicit solutions are easily obtained via the linearizing transformation (3.17*b*), which, in this particular case, becomes (see (3.18))

$$\psi_j(x,t)=\frac{\varphi_j(x,t)}{(\phi(x,t))^{1/2}}\qquad j=1,2 \tag{4.1a}$$

$$\phi(x,t)=1+2\int_{-\infty}^{x}\mathrm{d}x'(c_1|\varphi_1(x',t)|^2+c_2|\varphi_2(x',t)|^2) \tag{4.1b}$$

where $\varphi_j(x,t)$ $(j=1,2)$ is a solution of the Schrödinger equation

$$\mathrm{i}\varphi_{j,t}+a_j\varphi_{j,xx}=0. \tag{4.2}$$

General (discrete spectrum) solutions of (4.2) can be written as

$$\varphi_1(x,t)=\sum_{k=1}^{n}\exp[\tilde{a}_{1k}(x,t)+\mathrm{i}b_{1k}(x,t)] \tag{4.3a}$$

$$\varphi_2(x,t)=\sum_{k=1}^{m}\exp[\tilde{a}_{2k}(x,t)+\mathrm{i}b_{2k}(x,t)] \tag{4.3b}$$

$$\tilde{a}_{jk}(x,t)=p_{jk}(x-v_{jk}t)+\ln|A_{jk}| \tag{4.4}$$

$$b_{jk}(x,t)=\frac{v_{jk}}{2a_j}(x-w_{jk}t)+\alpha_{jk} \tag{4.5}$$

$$A_{jk}=|A_{jk}|\exp(\mathrm{i}\alpha_{jk}) \tag{4.6}$$

$$w_{jk}=\tfrac{1}{2}v_{jk}-2(a_jp_{jk})^2/v_{jk}. \tag{4.7}$$

Here the A_{jk} are $n+m$ arbitrary complex constants; the v_{jk} are $n+m$ arbitrary *real* parameters and the p_{jk} are $n+m$ arbitrary *positive* parameters, labelled in increasing order

$$0<p_{11}\leqslant p_{12}\cdots\leqslant p_{1n} \tag{4.8a}$$

$$0<p_{21}\leqslant p_{22}\cdots\leqslant p_{2m}. \tag{4.8b}$$

We note that the choice $p_{jk}>0$ in (4.8) entails that both $\varphi_j(x,t)$ and $\psi_j(x,t)$ vanish asymptotically as $x\to-\infty$, hence it guarantees convergence of the integral in definition (4.1*b*). We also note that solution (4.1*a*) may be singular at the zeros of the denominator $\phi(x,t)$, see (4.1). In order to avoid singularities, it is sufficient to assume, as we do hereafter, that $c_1>0$ and $c_2>0$.

The function $\phi(x,t)$ in (4.1a) can now be evaluated via (4.1b), (4.3) and (4.4)–(4.7). It has the form

$$\phi(x,t) = 1 + 2c_1\Bigg[\sum_{k=1}^{n}\frac{1}{2p_{1k}}\exp[2\tilde{a}_{1k}(x,t)] + 2\sum_{k=2}^{n}\sum_{\ell=1}^{k-1}\Bigg[(p_{1k}+p_{1\ell})^2 + \left(\frac{v_{1k}-v_{1\ell}}{2a_1}\right)^2\Bigg]^{-1/2}\exp[\tilde{a}_{1k}(x,t)+\tilde{a}_{1\ell}(x,t)]\cos(b_{1k}(x,t) - b_{1\ell}(x,t) + \beta_{k\ell}^{(1)})\Bigg] + 2c_2\Bigg[\sum_{k=1}^{m}\frac{1}{2p_{2k}}\exp[2\tilde{a}_{2k}(x,t)] + 2\sum_{k=2}^{m}\sum_{\ell=1}^{k-1}\Bigg[(p_{2k}+p_{2\ell})^2 + \left(\frac{v_{2k}-v_{2\ell}}{2a_2}\right)^2\Bigg]^{-1/2}\exp[\tilde{a}_{2k}(x,t)+\tilde{a}_{2\ell}(x,t)] \times \cos(b_{2k}(x,t) - b_{2\ell}(x,t) + \beta_{k\ell}^{(2)})\Bigg] \tag{4.9}$$

with

$$\beta_{k\ell}^{(j)} = \arctan\left[\frac{1}{2a_j}\frac{(v_{jk}-v_{j\ell})}{(p_{jk}+p_{j\ell})}\right] \qquad j = 1, 2. \tag{4.10}$$

In the following we concentrate our attention on the behaviour of the squared modulus of $\psi_j(x,t)$, which has the expression

$$|\psi_j(x,t)|^2 = \frac{|\varphi_j(x,t)|^2}{\phi(x,t)} \qquad j = 1, 2 \tag{4.11}$$

with $\phi(x,t)$ given by (4.9) and

$$|\varphi_1(x,t)|^2 = \sum_{k=1}^{n}\exp(2\tilde{a}_{1k}(x,t)) + 2\sum_{k=2}^{n}\sum_{\ell=1}^{k-1}\exp(\tilde{a}_{1k}(x,t)+\tilde{a}_{1\ell}(x,t)). \times \cos(b_{1k}(x,t) - b_{1\ell}(x,t)) \tag{4.12a}$$

$$|\varphi_2(x,t)|^2 = \sum_{k=1}^{m}\exp(2\tilde{a}_{2k}(x,t) + 2\sum_{k=2}^{m}\sum_{\ell=1}^{k-1}\exp(\tilde{a}_{2k}(x,t)+\tilde{a}_{2\ell}(x,t)). \times \cos(b_{2k}(x,t) - b_{2\ell}(x,t)). \tag{4.12b}$$

We assume for simplicity $p_{1n} > p_{1n-1}$ and $p_{2m} > p_{2m-1}$, and consider first the case $p_{1n} > p_{2m}$. The above formulae then imply that $|\psi_1(x,t)|^2$ has a kink profile

$$|\psi_1(x,t)|^2 \to \begin{cases} p_{1n}/c_1 & \text{as } x \to +\infty \\ 0 & \text{as } x \to -\infty \end{cases} \tag{4.13}$$

while $|\psi_2(x,t)|^2$ is localized

$$|\psi_2(x,t)|^2 \to 0 \qquad \text{as } x \to \pm\infty. \tag{4.14}$$

Of course in the opposite case, $p_{1n} < p_{2m}$, $|\psi_1(x,t)|^2 \to 0$ as $x \to \pm\infty$ (localized profile) while $|\psi_2(x,t)|^2 \to p_{2m}/c_2$ as $x \to +\infty$ (kink profile).

If instead $p_{1n} = p_{2m} = p$, both $|\psi_1(x,t)|^2$ and $|\psi_2(x,t)|^2$ tend to time-dependent values as $x \to +\infty$. Indeed (4.11), (4.9) and (4.12) then imply

$$|\psi_1(x,t)|^2 \xrightarrow[x\to+\infty]{} p|A_{1n}|^2/\{c_1|A_{1n}|^2 + c_2|A_{2m}|^2\exp[2p(v_{1n}-v_{2m})t]\} \tag{4.15a}$$

$$|\psi_2(x,t)|^2 \xrightarrow[x\to+\infty]{} p|A_{2m}|^2/\{c_2|A_{2m}|^2 + c_1|A_{1n}|^2\exp[2p(v_{2m}-v_{1n})t]\}. \tag{4.15b}$$

Note that these asymptotic expressions become constant if $v_{1n} = v_{2m}$.

If we assume $v_{1n} > v_{2m}$, (4.15*a*) and (4.15*b*) imply that in the remote past $(t \to -\infty)$ $|\psi_1(\infty, t)|^2$ tends to the constant value p/c_1, while $|\psi_2(\infty, t)|^2$ vanishes exponentially; in the future $(t \to +\infty)$, $|\psi_1(\infty, t)|^2$ vanishes exponentially while $|\psi_2(\infty, t)|^2$ tends to p/c_2. We have of course an opposite behaviour in the case $v_{1n} < v_{2m}$.

Next we show that in the simplest case, $n = m = 1$ (we drop the index k in (4.9) and (4.12)), with the assumption $p_1 = p_2 = p > 0$, the wave profiles $(|\psi_j(x, t)|^2)_x$ exhibit a boomeronic behaviour [12]. To this aim it is convenient to set $v_1 = -v_2 = v > 0$. With these parameters the expressions (4.11), with (4.12) and (4.9), specialize to

$$|\psi_j(x, t)|^2 = \frac{p}{2c_j} \exp[2pz_j(t)]\{1 + \mathrm{tgh}[p(x - \xi_j(t))]\} \qquad j = 1, 2 \quad (4.16)$$

with the following notation:

$$\xi_1(t) = x_1 + z_1(t) + vt \qquad \xi_2(t) = x_2 + z_2(t) - vt \qquad (4.17a)$$

$$x_j = -\frac{1}{2p} \ln(c_j|A_j|^2/p) \qquad j = 1, 2 \qquad (4.17b)$$

$$z_1(t) = -\frac{1}{2p} \ln\{1 + \exp[2p(x_1 - x_2 + 2vt)]\} \qquad (4.17c)$$

$$z_2(t) = -\frac{1}{2p} \ln\{1 + \exp[-2p(x_1 - x_2 + 2vt)]\}. \qquad (4.17d)$$

These simple explicit expressions need few comments. The charge density $|\psi_j|^2$ has a kink profile whose jump, $|\psi_j(+\infty, t)|^2 - |\psi_j(-\infty, t)|^2$, evolves from p/c_1 to zero as t goes from $-\infty$ to $+\infty$ for $j = 1$, and instead from zero to p/c_2 for $j = 2$. A simpler picture obtains, by looking at the x-derivatives, namely at the expression

$$(|\psi_j(x, t)|^2)_x = \frac{p^2}{2c_j} \frac{\exp[2pz_j(t)]}{\cosh^2[p(x - \xi_j(t))]} \qquad j = 1, 2. \qquad (4.18)$$

Indeed, it is easily found that the asymptotic velocities $\dot{\xi}_j = \mathrm{d}\xi_j/\mathrm{d}t$ of the solitary profile of both the upper ($j = 1$) and lower ($j = 2$) component have opposite signs at $t = \pm\infty$, namely

$$\dot{\xi}_j(t) \longrightarrow \mp v \qquad t \to \pm\infty \qquad j = 1, 2\ . \qquad (4.19)$$

This displays the boomeronic character [12] of the motion of these solitons. Moreover, the peak amplitudes of the solitary waves (4.18) for $j = 1$ and $j = 2$ evolve in time in opposite ways, namely

$$(|\psi_1(\xi_1(t), t)|^2)_x \to \begin{cases} \dfrac{p^2}{2c_1} & t \to -\infty \\ 0 & t \to +\infty \end{cases} \qquad (4.20a)$$

$$(|\psi_2(\xi_2(t), t)|^2)_x \to \begin{cases} 0 & t \to -\infty \\ \dfrac{p^2}{2c_2} & t \to +\infty. \end{cases} \qquad (4.20b)$$

Appendix. Derivation of (3.17*b*) from (3.17*a*)

Let us start from the relation

$$\rho(x, t) = R(x, t) \exp\left[2\int_{-\infty}^{x} \mathrm{d}x'\, R(x', t)\right] \qquad (A.1)$$

which is implied by (3.2), (3.7) and (3.17*a*). This implies

$$\rho_x/\rho = R_x/R + 2R \tag{A.2a}$$

as well as

$$\lim_{x\to-\infty}[\rho(x,t)/R(x,t)] = 1. \tag{A.2b}$$

Now set

$$\psi = f\varphi \tag{A.3}$$

hence (via (3.2) and (3.7))

$$R = f^2\rho \tag{A.4}$$

hence (via (A.2*a*) and (A.4))

$$f_x f^{-3} = -\rho \tag{A.5a}$$

and (via (A.2*b*))

$$f(-\infty, t) = 1. \tag{A.5b}$$

Hence

$$f(x,t) = \left[1 + 2\int_{-\infty}^{x} \mathrm{d}x'\rho(x',t)\right]^{-1/2}. \tag{A.6}$$

Via (A.3) and (3.2) this yields (3.17*b*).

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